ASYMPTOTICS FOR THE SMALL FRAGMENTS OF THE FRAGMENTATION AT NODES

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Abstract. We consider the fragmentation at nodes of the Lévy continuous random tree introduced in a previous paper. In this framework we compute the asymptotic for the number of small fragments at time θ. This limit is increasing in θ and discontinuous. In the α-stable case the fragmentation is self-similar with index $1/\alpha$, with $\alpha \in (1,2)$ and the results are close to those Bertoin obtained for general self-similar fragmentations but with an additional assumtion which is not fulfilled here.

1. Introduction

A fragmentation process is a Markov process which describes how an object with given total mass evolves as it breaks into several fragments randomly as time passes. Notice there may be loss of mass but no creation. Those processes have been widely studied in the recent years, see Bertoin [7] and references therein. To be more precise, the state space of a fragmentation process is the set of the non-increasing sequences of masses with finite total mass:

$$\mathcal{S}^{\downarrow} = \left\{ s = (s_1, s_2, \ldots);\ s_1 \geq s_2 \geq \cdots \geq 0 \quad \text{and} \quad \Sigma(s) = \sum_{k=1}^{+\infty} s_k < +\infty \right\}.$$

If we denote by P_s the law of a $\mathcal{S}^{\downarrow}$-valued process $\Lambda = (\Lambda(\theta), \theta \geq 0)$ starting at $s = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}$, we say that Λ is a fragmentation process if it is a Markov process such that $\theta \mapsto \Sigma(\Lambda(\theta))$ is non-increasing and if it fulfills the fragmentation property: the law of $(\Lambda(\theta), \theta \geq 0)$ under P_s is the non-increasing reordering of the fragments of independent processes of respective laws $P_{(s_1,0,\ldots)}, P_{(s_2,0,\ldots)}, \ldots$. In other words, each fragment after dislocation behaves independently of the others, and its evolution depends only on its initial mass. As a consequence, to describe the law of the fragmentation process with any initial condition, it suffices to study the laws $P_r := P_{(r,0,\ldots)}$ for any $r \in (0, +\infty)$, i.e. the law of the fragmentation process starting with a single mass r.

A fragmentation process is said to be self-similar of index α' if, for any $r > 0$, the process Λ under P_r is distributed as the process $(r\Lambda(r^{\alpha'}\theta), \theta \geq 0)$ under P_1. Bertoin [5] proved that the law of a self-similar fragmentation is characterized by: the index of self-similarity α', an erosion coefficient c which corresponds to a rate of mass loss, and a dislocation measure ν on $\mathcal{S}^{\downarrow}$ which describes sudden dislocations of a fragment of mass 1. The dislocation measure of a fragment of size r, ν_r is given by $\int F(s)\nu_r(ds) = r^{\alpha'} \int F(rs)\nu(ds)$.

When there is no loss of mass (which implies that $c = 0$ and $\alpha' > 0$), under some additional assumptions, the number of fragments at a fixed time is infinite. A natural question is therefore to study the asymptotic behavior when ε goes down to 0 of $N^{\varepsilon}(\theta) = \text{Card}\ \{i, \Lambda_i(\theta) >$

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$\varepsilon\}$ where $\Lambda(\theta) = (\Lambda_1(\theta), \Lambda_2(\theta), \ldots)$ is the state of the fragmentation at time θ, see Bertoin [6] and also Haas [10] when α' is negative.

The goal of this paper is to study the same problem for the fragmentation at nodes of the Lévy continuous random tree constructed in [1].

In [12] and [11], Le Gall and Le Jan associated to a Lévy process with no negative jumps that does not drift to infinity, $X = (X_s, s \geq 0)$ with Laplace exponent ψ, a continuous state branching process (CSBP) and a Lévy continuous random tree (CRT) which keeps track of the genealogy of the CSBP. The Lévy CRT can be coded by the so called height process, $H = (H_s, s \geq 0)$. Informally H_s gives the distance (which can be understood as the number of generations) between the individual labeled s and the root, 0, of the CRT. The precise definition of ψ we consider is given at the beginning of Section 2.1.

The ideas of [1] in order to construct a fragmentation process from this CRT is to mark the nodes of the tree in a Poissonian manner. We then cut the CRT at these marked nodes and the "sizes" of the resulting subtrees give the state of the fragmentation at some time. As time θ increases, the parameter of the Poisson processes used to mark the nodes increases as well as the set of the marked nodes. This gives a fragmentation process with no loss of mass. When the initial Lévy process is stable i.e. when $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2]$, the fragmentation is self-similar with index $1/\alpha$ and with a zero erosion coefficient, see also see [2] and [4] for $\alpha = 2$, or [13] for $\alpha \in (1, 2)$. For a general sub-critical or critical CRT, there is no more scaling property, and the dislocation measure, which describes how a fragment of size $r > 0$ is cut in smaller pieces, cannot be expressed as a nice function of the dislocation measure of a fragment of size 1. In [1], the authors give the family of dislocation measures $(\nu_r, r > 0)$ for the fragmentation at node of a general sub-critical or critical CRT. Intuitively ν_r describes the way a mass r breaks in smaller pieces.

We denote by $\mathbb{N}$ the excursion measure of the Lévy process X (the fragmentation process is then defined under this measure). We denote by σ the length of the excursion. We have (see Section 3.2.2. in [9]) that

$$\mathbb{N}[1 - \mathrm{e}^{-\lambda\sigma}] = \psi^{-1}(\lambda), \tag{1}$$

and ψ^{-1} is the Laplace exponent of a subordinator (see [3], chap. VII), whose Lévy measure we denote by π_*. The distribution of σ under $\mathbb{N}$ is given by π_*. As π_* is a Lévy measure, we have $\int_{(0,\infty)} (1 \wedge r)\, \pi_*(dr) < \infty$. For $\varepsilon > 0$, we write

$$\bar{\pi}_*(\varepsilon) = \pi_*((\varepsilon, \infty)) = \mathbb{N}[\sigma > \varepsilon] \quad \text{and} \quad \varphi(\varepsilon) = \int_{(0,\varepsilon]} r\pi_*(dr) = \mathbb{N}[\sigma \mathbf{1}_{\{\sigma \leq \varepsilon\}}].$$

If $\Lambda(\theta) = (\Lambda_1(\theta), \Lambda_2(\theta), \ldots)$ is the state of the fragmentation at time θ, we denote by $N^\varepsilon(\theta)$ the number of fragments of size greater than ε i.e.

$$N^\varepsilon(\theta) = \sum_{k=1}^{+\infty} \mathbf{1}_{\{\Lambda_k(\theta) > \varepsilon\}} = \sup\{k \geq 1, \Lambda_k(\theta) > \varepsilon\}$$

with the convention $\sup \emptyset = 0$. And we denote by $M^\varepsilon(\theta)$ the mass of the fragments of size less than ε i.e.

$$M^\varepsilon(\theta) = \sum_{k=1}^{+\infty} \Lambda_k(\theta) \mathbf{1}_{\{\Lambda_k(\theta) \leq \varepsilon\}} = \sum_{k=N^\varepsilon(\theta)+1}^{+\infty} \Lambda_k(\theta).$$

Let $\mathcal{J} = \{s \geq 0,\ X_s > X_{s-}\}$ and let $(\Delta_s,\ s \in \mathcal{J})$ be the set of jumps of X. Conditionally on $(\Delta_s, s \in \mathcal{J})$, let $(T_s, s \in \mathcal{J})$ be a family of independant random variables, such that T_s has exponential distribution with mean $1/\Delta_s$. T_s is the time at which the node of the CRT

associated to the jump Δ_s is marked in order to construct the fragmentation process. Under $\mathbb{N}$, we denote by $R(\theta)$ the mass of the marked nodes of the Lévy CRT i.e.

$$R(\theta) = \sum_{s \in \mathcal{J} \cap [0,\sigma]} \Delta_s \mathbf{1}_{\{T_s \leq \theta\}}.$$

The main result of this paper is then the following Theorem.

Theorem 1.1. *We have* $\lim_{\varepsilon \to 0} \frac{N^\varepsilon(\theta)}{\bar{\pi}_*(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{M^\varepsilon(\theta)}{\varphi(\varepsilon)} = R(\theta)$ *in* $L^2\big(\mathbb{N}[\mathrm{e}^{-\beta\sigma}\,\cdot]\big)$, *for any* $\beta > 0$.

We consider the stable case $\psi(\lambda) = \lambda^\alpha$, where $\alpha \in (1,2)$. We have

$$\pi_*(dr) = (\alpha\Gamma(1-\alpha^{-1}))^{-1} r^{-1-1/\alpha}\, dr,$$

which gives

$$\bar{\pi}_*(\varepsilon) = \Gamma(1-\alpha^{-1})^{-1}\varepsilon^{-1/\alpha}, \qquad \text{and} \qquad \varphi(\varepsilon) = \Big((\alpha-1)\Gamma(1-\alpha^{-1})\Big)^{-1}\varepsilon^{1-\alpha^{-1}}.$$

From scaling property, there exists a version of $(\mathbb{N}_r, r > 0)$ such that for all $r > 0$ we have $\mathbb{N}_r[F((X_s, s \in [0,r]))] = \mathbb{N}_1[F((r^{1/\alpha}X_{s/r}, s \in [0,r]))]$ for any non-negative measurable function F defined on the set of càd-làg paths.

Proposition 1.2. *Let* $\psi(\lambda) = \lambda^\alpha$, *for* $\alpha \in (1,2)$. *For all* $\theta > 0$, *we have* $\mathbb{N}$*.a.e or* $\mathbb{N}_1$*-a.s.*

$$\lim_{\varepsilon \to 0} \Gamma(1-1/\alpha)\varepsilon^{1/\alpha} N^\varepsilon(\theta) = \lim_{\varepsilon \to 0} (\alpha-1)\Gamma(1-1/\alpha)\frac{M^\varepsilon(\theta)}{\varepsilon^{1-1/\alpha}} = R(\theta). \tag{2}$$

Remark 1.3. Notice the similarity with the results in [8] on asymptotics for the small fragments in case of the fragmentation at height of the CRT: the local time of the height process is here replaced by the functional R.

Remark 1.4. Let us compare the result of Proposition 1.2 with the main Theorem of [6], which we recall now. Let Λ be a self-similar fragmentation of index $\alpha > 0$, erosion coefficient $c = 0$ and dislocation measure ν. We set

$$\varphi_b(\varepsilon) = \int_{\mathcal{S}^\downarrow} (\sum_{i=1}^\infty \mathbf{1}_{\{x_i > \varepsilon\}} - 1)\nu_1(dx),$$

$$f_b(\varepsilon) = \int_{\mathcal{S}^\downarrow} \sum_{i=1}^\infty x_i \mathbf{1}_{\{x_i < \varepsilon\}} \nu_1(dx),$$

$$g_b(\varepsilon) = \int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^\infty x_i \mathbf{1}_{\{x_i < \varepsilon\}}\right)^2 \nu_1(dx).$$

If there exists $\beta \in (0,1)$ such that φ_b is regularly varying at 0 with index $-\beta$ (which is equivalent to f_b is regularly varying at 0 with index $1-\beta$), and if there exists two positive constants c, η such that

$$g_b(\varepsilon) \leq c f_b^2(\varepsilon)(\log 1/\varepsilon)^{-(1+\eta)}, \tag{3}$$

then a.s.

$$\lim_{\varepsilon \to 0} \frac{N^\varepsilon(\theta)}{\varphi_b(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{M^\varepsilon(\theta)}{f_b(\varepsilon)} = \int_0^\theta \sum_{i=1}^\infty \Lambda_i(u)^{\alpha+\beta} du.$$

In our case, we have φ and $\bar{\pi}_*$ equivalent to φ_b and f_b (up to multiplicative constants, see Lemmas 5.1 and 5.2). The normalizations are consequently the same. However, we have

here $g_b(\varepsilon) = O(f_b^2(\varepsilon))$ (see Lemma 5.3) and Bertoin's assumption (3) is not fulfilled. When this last assumption holds, remark the limit process is an increasing continuous process (as θ varies). In our case this assumption does not hold and the limit process $(R(\theta), \theta \geq 0)$ is still increasing but discontinuous as $R(\theta)$ is a pure jump process (this is an increasing sum of marked masses).

The paper is organized as follows. In Section 2, we recall the definition and properties of the height and exploration processes that code the Lévy CRT and we recall the construction of the fragmentation process associated to the CRT. The proofs of Theorem 1.1 and Proposition 1.2 are given in Section 3. Notice computations given in the proof of Lemma 3.1 based on Propositions 2.2 and 2.3 are enough to characterize the transition kernel of the fragmentation Λ. We characterize the law of the scaling limit $R(\theta)$ in Section 4. The computation needed for Remark 1.4 are given in Section 5.

2. Notations

2.1. **The exploration process.** Let ψ denote the Laplace exponent of X: $\mathbb{E}\left[\mathrm{e}^{-\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)}$, $\lambda > 0$. We shall assume there is no Brownian part, so that

$$\psi(\lambda) = \alpha_0\lambda + \int_{(0,+\infty)} \pi(d\ell) \left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\right],$$

with $\alpha_0 \geq 0$ and the Lévy measure π is a positive σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi(d\ell) < \infty$. Following [9], we shall also assume that X is of infinite variation a.s. which implies that $\int_{(0,1)} \ell\pi(d\ell) = \infty$. Notice those hypothesis are fulfilled in the stable case: $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2)$. For $\lambda \geq 1/\varepsilon > 0$, we have $\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \geq \frac{1}{2}\lambda\ell\mathbf{1}_{\{\ell\geq 2\varepsilon\}}$, which implies that $\lambda^{-1}\psi(\lambda) \geq \alpha_0 + \int_{(2\varepsilon,\infty)} \ell\, \pi(d\ell)$. We deduce that

$$\lim_{\lambda\to\infty} \frac{\lambda}{\psi(\lambda)} = 0. \tag{4}$$

The so-called exploration process $\rho = (\rho_t, t \geq 0)$ is Markov process taking values in $\mathcal{M}_f$, the set of positive measures on $\mathbb{R}_+$. The height process at time t is defined as the supremum of the closed support of ρ_t (with the convention that $H_t = 0$ if $\rho_t = 0$). Informally, H_t gives the distance (which can be understood as the number of generations) between the individual labeled t and the root, 0, of the CRT. In some sense $\rho_t(dv)$ records the "number" of brothers, with labels larger than t, of the ancestor of t at generation v.

We recall the definition and properties of the exploration process which are given in [12], [11] and [9]. The results of this section are mainly extracted from [9].

Let $I = (I_t, t \geq 0)$ be the infimum process of X, $I_t = \inf_{0\leq s\leq t} X_s$. We will also consider for every $0 \leq s \leq t$ the infimum of X over $[s, t]$:

$$I_t^s = \inf_{s\leq r\leq t} X_r.$$

There exists a sequence $(\varepsilon_n, n \in \mathbb{N}^*)$ of positive real numbers decreasing to 0 s.t.

$$\tilde{H}_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t \mathbf{1}_{\{X_s < I_t^s + \varepsilon_k\}}\, ds$$

exists and is finite a.s. for all $t \geq 0$.

The point 0 is regular for the Markov process $X - I$, $-I$ is the local time of $X - I$ at 0 and the right continuous inverse of $-I$ is a subordinator with Laplace exponent ψ^{-1} (see [3], chap. VII). Notice this subordinator has no drift thanks to (4). Let π_* denote the corresponding

Lévy measure. Let $\mathbb{N}$ be the associated excursion measure of the process $X - I$ out of 0, and $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under $\mathbb{N}$. Under $\mathbb{N}$, $X_0 = I_0 = 0$.

For $\mu \in \mathcal{M}_f$, we define $H^\mu = \sup\{x \in \text{supp } \mu\}$, where supp μ is the closed support of the measure μ. From Section 1.2 in [9], there exists a $\mathcal{M}_f$-valued process, $\rho^0 = (\rho^0_t, t \geq 0)$, called the exploration process, such that:

- A.s., for every $t \geq 0$, we have $\langle \rho^0_t, 1\rangle = X_t - I_t$, and the process ρ^0 is càd-làg.
- The process $(H^0_s = H^{\rho^0_s}, s \geq 0)$ taking values in $[0, \infty]$ is lower semi-continuous.
- For each $t \geq 0$, a.s. $H^0_t = \tilde{H}_t$.
- For every measurable non-negative function f defined on $\mathbb{R}_+$,

$$\langle \rho^0_t, f\rangle = \int_{[0,t]} f(H^0_s)\, d_s I^s_t,$$

  or equivalently, with δ_x being the Dirac mass at x,

$$\rho^0_t(dr) = \sum_{\substack{0<s\leq t\\ X_{s-}<I^s_t}} (I^s_t - X_{s-})\delta_{H^0_s}(dr).$$

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ a.s. To get the Markov property of ρ, we must define the process ρ started at any initial measure $\mu \in \mathcal{M}_f$. For $a \in [0, \langle \mu, 1\rangle]$, we define the erased measure $k_a\mu$ by

$$k_a\mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1\rangle - a), \quad \text{for } r \geq 0.$$

If $a > \langle \mu, 1\rangle$, we set $k_a\mu = 0$. In other words, the measure $k_a\mu$ is the measure μ erased by a mass a backward from H^μ.

For $\nu, \mu \in \mathcal{M}_f$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f$ of the two measures by:

$$\big\langle [\mu, \nu], f\big\rangle = \big\langle \mu, f\big\rangle + \big\langle \nu, f(H^\mu + \cdot)\big\rangle,$$

for f non-negative measurable. Eventually, we set for every $\mu \in \mathcal{M}_f$ and every $t > 0$,

$$\rho_t = \big[k_{-I_t}\mu, \rho^0_t\big].$$

We say that $\rho = (\rho_t, t \geq 0)$ is the process ρ started at $\rho_0 = \mu$, and write $\mathbb{P}_\mu$ for its law. We set $H_t = H^{\rho_t}$. The process ρ is càd-làg (with respect to the weak convergence topology on $\mathcal{M}_f$) and strong Markov.

2.2. **Notations for the fragmentation at nodes.** We recall the construction of the fragmentation under $\mathbb{N}$ given in [1] in an equivalent but easier way to understand. Recall $(\Delta_s, s \in \mathcal{J})$ is the set of jumps of X and T_s is the time at which the jump Δ_s is marked. Conditionally on $(\Delta_s, s \in \mathcal{J})$, $(T_s, s \in \mathcal{J})$ is a family of independent random variables, such that T_s has exponential distribution with mean $1/\Delta_s$. We consider the family of measures (increasing in θ) defined for $\theta \geq 0$ and $t \geq 0$ by

$$\tilde{m}^\theta_t(dr) = \sum_{\substack{0<s\leq t\\ X_{s-}<I^s_t}} \mathbf{1}_{\{T_s\leq\theta\}}\, \delta_{H_s}(dr).$$

Intuitively, $\tilde{m}^\theta_t$ describes the marked masses of the measure ρ_t i.e. the marked nodes of the associated CRT.

Then we cut the CRT according to these marks to obtain the state of the fragmentation process at time θ. To construct the fragmentation, let us consider the following equivalence relation $\mathcal{R}^\theta$ on $[0, \sigma]$, defined under $\mathbb{N}$ or $\mathbb{N}_\sigma$ by

$$s\mathcal{R}^\theta t \iff \tilde{m}_s^\theta\big([H_{s,t}, H_s]\big) = \tilde{m}_t^\theta\big([H_{s,t}, H_t]\big) = 0, \tag{5}$$

where $H_{s,t} = \inf_{u\in[s,t]} H_u$. Intuitively, two points s and t belongs to the same class of equivalence (i.e. the same fragment) at time θ if there is no cut on their lineage down to their most recent common ancestor, that is $\tilde{m}_s^\theta$ put no mass on $[H_{s,t}, H_s]$ nor $\tilde{m}_t^\theta$ on $[H_{s,t}, H_t]$. Notice cutting occurs on branching points, that is at node of the CRT. Each node of the CRT correspond to a jump of the underlying Lévy process X. The fragmentation process at time θ is then the Lebesgue measures (ranked in non-increasing order) of the equivalent classes of $\mathcal{R}^\theta$.

Remark 2.1. In [1], see definition (14), we use another family of measures $m_t^{(\theta)}$. From their construction, notice that $\tilde{m}_t^\theta$ is absolutely continuous w.r.t. $m_t^{(\theta)}$ and $m_t^{(\theta)}$ is absolutely continuous w.r.t. $\tilde{m}_t^\theta$, if we take $T_s = \inf\{V_{s,u}, u > 0\}$, where $\sum_{u>0} \delta_{V_{s,u}}$ is a Poisson point measure on $\mathbb{R}_+$ with intensity $\Delta_s \mathbf{1}_{\{u>0\}}$, see Section 3.1 in [1]. In particular $\tilde{m}_t^\theta$ and $m_t^{(\theta)}$ define the same equivalence relation and therefore the same fragmentation.

In order to index the fragments, we define the "generation" of a fragment. For any $s \leq \sigma$, let us define $H_s^0 = 0$ and recursively for $k \in \mathbb{N}$,

$$H_s^{k+1} = \inf\Big\{u \geq 0, \tilde{m}_s^\theta\big((H_s^k, u]\big) > 0\Big\},$$

with the usual convention $\inf \emptyset = +\infty$. We set the "generation" of s as

$$K_s = \sup\{j \in \mathbb{N},\ H_s^j < +\infty\}.$$

Notice that if $s\mathcal{R}^\theta t$, then $K_s = K_t$. In particular all elements of a fragment have the same "generation". We also call this "generation" the "generation" of the fragment. Let $(\sigma^{i,k}(\theta), i \in I_k)$ be the family of lengths of fragments in "generation" k. Notice that I_0 is reduced to one point, say 0, and we write

$$\tilde{\sigma}(\theta) = \sigma^{0,0}(\theta)$$

for the fragment which contains the root. The joint law of $(\tilde{\sigma}(\theta), \sigma)$ is given in Proposition 7.3 in [1].

Let $(r^{j,k+1}(\theta), j \in J_{k+1})$ be the family of sizes of the marked nodes attached to the snake of "generation" k. More precisely,

$$\big\{r^{j,k+1}(\theta), j \in J_{k+1}\big\} = \big\{\Delta_s, T_s \leq \theta \quad \text{and} \quad K_s = k+1\big\}.$$

We set, for $k \in \mathbb{N}$,

$$L_k(\theta) = \sum_{i\in I_k} \sigma^{i,k}(\theta), \quad N_k^\varepsilon(\theta) = \sum_{i\in I_k} \mathbf{1}_{\{\sigma^{i,k}(\theta)>\varepsilon\}}, \quad M_k^\varepsilon(\theta) = \sum_{i\in I_k} \sigma^{i,k}(\theta)\mathbf{1}_{\{\sigma^{i,k}\leq\varepsilon\}},$$

and we set, for $k \in \mathbb{N}^*$,

$$R_k(\theta) = \sum_{j\in J_k} r^{i,k}(\theta).$$

We set $R_0 = 0$. Let us remark that we have $\sigma = \sum_{k\geq 0} L_k(\theta)$, $N^\varepsilon(\theta) = \sum_{k\geq 0} N_k^\varepsilon(\theta)$, $M^\varepsilon(\theta) = \sum_{k\geq 0} M_k^\varepsilon(\theta)$ and $R(\theta) = R_k(\theta)$.

Let $\mathcal{F}_k$ be the σ-field generated by $((\sigma^{i,l}(\theta), i \in I_l), R_l(\theta))_{0\leq l\leq k}$. As a consequence of the special Markov property (Theorem 5.2 of [1]) and using the recursive construction of Lemma 8.6 of [1], we have the following Propositions.

Proposition 2.2. *Under $\mathbb{N}$, conditionally on $\mathcal{F}_{k-1}$ and $R_k(\theta)$, $\sum_{i\in I_k} \delta_{\sigma^{i,k}(\theta)}$ is distributed as a Poisson point process with intensity $R_k(\theta)\mathbb{N}[d\tilde{\sigma}(\theta)]$.*

Proposition 2.3. *Under $\mathbb{N}$, conditionally on $\mathcal{F}_{k-1}$, $\sum_{j\in J_k} \delta_{r^{j,k}(\theta)}$ is distributed as a Poisson point process with intensity $L_{k-1}(\theta)(1-\mathrm{e}^{-\theta r})\,\pi(dr)$.*

Remark 2.4. Those Propositions allow to compute the law of the fragmentation $\Lambda(\theta)$ for a given θ (see computations of Laplace transform in the proof of Lemma 3.1.

Let us recall that the key object in [1] is the tagged fragment which contains the root. Recall its size is denoted by $\tilde{\sigma}(\theta)$. This fragment corresponds to the subtree of the initial CRT (after pruning) that contains the root. This subtree is a Lévy CRT and the Laplace exponent of the associated Lévy process is

$$\psi_\theta(\lambda) := \psi(\lambda+\theta) - \psi(\theta), \quad \lambda \geq 0.$$

This implies $\psi_\theta^{-1}(v) = \psi^{-1}(v+\psi(\theta)) - \theta$ and we deduce from (4)

$$\lim_{\lambda\to\infty} \psi_\theta^{-1}(\lambda)/\lambda = 0. \tag{6}$$

We also have (see (3) for the first equality with ψ replaced by ψ_θ)

$$\mathbb{N}\left[1-\mathrm{e}^{-\beta\tilde{\sigma}(\theta)}\right] = \psi_\theta^{-1}(\beta) \quad \text{and} \quad \mathbb{N}\left[\tilde{\sigma}(\theta)\,\mathrm{e}^{-\beta\tilde{\sigma}(\theta)}\right] = \frac{1}{\psi_\theta'(\psi_\theta^{-1}(\beta))}. \tag{7}$$

3. Proofs

We fix $\theta > 0$. As θ is fixed, we will omit to mention the dependence w.r.t. θ of the different quantities in this section: for example we write $\tilde{\sigma}$ and N^ε for $\tilde{\sigma}(\theta)$ and $N^\varepsilon(\theta)$. We set

$$\mathcal{N}^\varepsilon = N^\varepsilon - \mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} \qquad \text{and} \qquad \mathcal{M}^\varepsilon = M^\varepsilon - \tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}}.$$

3.1. **Proof of Theorem 1.1.** The poof is in four steps. In the first step we compute the Laplace transform of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R, \sigma)$. From there we could prove the convergence of Theorem 1.1 with a convergence in probability instead of in L^2. However we need a convergence speed to get the a.s. convergence in the α-stable case of Proposition 1.2. In the second step, we check the computed Laplace transform has the necessary regularity in order to derive in the third step the second moment of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R)$ under $\mathbb{N}[\mathrm{e}^{-\beta\sigma}\,\cdot]$. In the last step we check the convergence statement of the second moment.

In a **first step**, we give the joint law under $\mathbb{N}$ of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R, \sigma)$ by computing for $x > 0$, $y > 0$, $\beta > 0$, $\gamma > 0$,

$$\mathbb{N}\left[\mathrm{e}^{-(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R + \beta\sigma)}\,\middle|\tilde{\sigma}\right].$$

By monotone convergence, we have

$$\mathbb{N}\left[\mathrm{e}^{-(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R + \beta\sigma)}\,\middle|\tilde{\sigma}\right] = \lim_{n\to\infty} \mathbb{N}\left[\mathrm{e}^{-\left(\beta\tilde{\sigma} + \sum_{l=1}^n (xN_l^\varepsilon + yM_l^\varepsilon + \gamma R_l + \beta L_l)\right)}\,\middle|\tilde{\sigma}\right]. \tag{8}$$

We define the function $H_{(x,y,\gamma)}$ by

$$H_{(x,y,\gamma)}(c) = G\left(\gamma + \mathbb{N}\left[1 - \mathrm{e}^{-(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}} + c\tilde{\sigma})}\right]\right),$$

where for $a \geq 0$,

$$G(a) = \int \pi(dr) \left(1 - \mathrm{e}^{-\theta r}\right) \left(1 - \mathrm{e}^{-ar}\right) = \psi(\theta + a) - \psi(a) - \psi(\theta) = \psi_\theta(a) - \psi(a). \tag{9}$$

Recall $\mathcal{F}_k$ is the σ-field generated by $((\sigma^{i,l}, i \in I_l), R_l)_{0 \leq l \leq k}$. We then have the following Lemma.

Lemma 3.1. *For $x, y, \gamma \in \mathbb{R}_+$, $\varepsilon > 0$, we have for $k \in \mathbb{N}^*$,*

$$\mathbb{N}\left[\mathrm{e}^{-(xN_k^\varepsilon + yM_k^\varepsilon + cL_k + \gamma R_k)} \Big| \mathcal{F}_{k-1}\right] = \mathrm{e}^{-H_{(x,y,\gamma)}(c) L_{k-1}}.$$

Proof. As a consequence of Proposition 2.2, we have

$$\mathbb{N}\left[\mathrm{e}^{-(xN_k^\varepsilon + yM_k^\varepsilon + cL_k + \gamma R_k)} \Big| \mathcal{F}_{k-1}, R_k\right] = \mathrm{e}^{-R_k(\gamma + \mathbb{N}[1 - \exp(-(x\mathbf{1}_{\{\tilde\sigma > \varepsilon\}} + y\tilde\sigma \mathbf{1}_{\{\tilde\sigma \leq \varepsilon\}} + c\tilde\sigma))])}.$$

As a consequence of Proposition 2.3, we have

$$\mathbb{N}\left[\mathrm{e}^{-R_k(\gamma + \mathbb{N}[1 - \exp(-(x\mathbf{1}_{\{\tilde\sigma > \varepsilon\}} + y\tilde\sigma \mathbf{1}_{\{\tilde\sigma \leq \varepsilon\}} + c\tilde\sigma))])} \Big| \mathcal{F}_{k-1}\right] = \mathrm{e}^{-H_{(x,y,\gamma)}(c) L_{k-1}}.$$

□

We define the constants $c_{(k)}$ by induction:

$$c_{(0)} = 0 \qquad \text{and} \qquad c_{(k+1)} = H_{(x,y,\gamma)}(c_{(k)} + \beta).$$

An immediate backward induction yields (recall $L_0 = \tilde\sigma$): for every integer $n \geq 1$, we have

$$\mathbb{N}\left[\mathrm{e}^{-\left(\sum_{l=1}^n (xN_l^\varepsilon + yM_l^\varepsilon + \gamma R_l + \beta L_l)\right)} \Big| \tilde\sigma\right] = \mathrm{e}^{-c_{(n)}\tilde\sigma}.$$

Notice the function G is of class $\mathcal{C}^\infty$ on $(0, \infty)$, concave increasing and the function

$$c \mapsto \mathbb{N}\left[1 - \mathrm{e}^{-(x\mathbf{1}_{\{\tilde\sigma > \varepsilon\}} + y\tilde\sigma\mathbf{1}_{\{\tilde\sigma \leq \varepsilon\}} + (\beta + c)\tilde\sigma)}\right]$$

is of class $\mathcal{C}^\infty$ on $[0, \infty)$ and is concave increasing. This implies that $H_{(x,y,\gamma)}$ is concave increasing and of class $\mathcal{C}^\infty$. Notice that

$$x\mathbf{1}_{\{\tilde\sigma > \varepsilon\}} + y\tilde\sigma\mathbf{1}_{\{\tilde\sigma \leq \varepsilon\}} + c\tilde\sigma \leq (\frac{x}{\varepsilon} + y + c)\tilde\sigma.$$

In particular, we have $H_{(x,y,\gamma)}(c) \leq G(\gamma + \psi_\theta^{-1}(\frac{x}{\varepsilon} + y + c))$. As $\lim_{a \to \infty} G'(a) = 0$, this implies that $\lim_{a \to \infty} G(a)/a = 0$. Since $\lim_{\lambda \to \infty} \psi_\theta^{-1}(\lambda) = \infty$, we deduce thanks to (6) that

$$\lim_{c \to \infty} \frac{H_{(x,y,\gamma)}(c)}{c} = 0. \tag{10}$$

For $\gamma > 0$, notice $H_{(x,y,\gamma)}(0) > 0$. As the function $H_{(x,y,\gamma)}$ is increasing and continuous, we deduce the sequence $(c_{(n)}, n \geq 0)$ is increasing and converges to the unique root, say c', of $c = H_{(x,y,\gamma)}(c + \beta)$. And we deduce from (8) that

$$\mathbb{N}\left[\mathrm{e}^{-(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R + \beta\sigma)} \Big| \tilde\sigma\right] = \mathrm{e}^{-(\beta + c')\tilde\sigma}. \tag{11}$$

In a **second step**, we look at the dependency of the root of $c = H_{(x,y,\gamma)}(c + \beta)$ in (x, y, γ).

Let $\varepsilon, x, y, \beta, \gamma \in (0, \infty)$ be fixed. There exists $a > 0$ small enough such that for all $z \in (-a, a)$, we have $z\gamma + \mathbb{N}[1 - \mathrm{e}^{-\beta\tilde\sigma/2}] > 0$ and for all $\tilde\sigma \geq 0$,

$$z(x\mathbf{1}_{\{\tilde\sigma > \varepsilon\}} + y\tilde\sigma\mathbf{1}_{\{\tilde\sigma \leq \varepsilon\}}) + \beta\tilde\sigma \geq \beta\tilde\sigma/2.$$

We consider the function J defined on $(-\beta/2, \infty) \times (-a, a)$ by

$$J(c, z) = H_{zx,zy,z\gamma}(c + \beta) - c.$$

From the regularity of G, we deduce the function J is of class $\mathcal{C}^\infty$ on $(-\beta/2, \infty) \times (-a, a)$ and the function $c \mapsto J(c, z)$ is concave. Notice that $J(0, z) > 0$ for all $z \in (-a, a)$. This and (10) implies that there exists a unique solution $c(z)$ to the equation $J(c, z) = 0$ and that $\dfrac{\partial J}{\partial c}(c(z), z) < 0$ for all $z \in (-a, a)$. The implicit function Theorem implies the function $z \mapsto c(z)$ is of class $\mathcal{C}^\infty$ on $(-a, a)$. In particular, we have $c(z) = c_0 + zc_1 + \dfrac{z^2}{2}c_2 + o(z^2)$. We deduce from (11) that for all $z \in [0, a)$,

$$\mathbb{N}\left[\mathrm{e}^{-z(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R) - \beta\sigma} \middle| \tilde{\sigma}\right] = \mathrm{e}^{-(\beta + c(z))\tilde{\sigma}} = \mathrm{e}^{-(\beta + c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2))\tilde{\sigma}}.$$

In a **third step**, we investigate the second moment $\mathbb{N}\left[(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R)^2 \,\mathrm{e}^{-\beta\sigma}\right]$. Standard results on Laplace transforms, implies the second moment is finite and

$$\mathbb{N}\left[(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R)^2 \,\mathrm{e}^{-\beta\sigma} \middle| \tilde{\sigma}\right] = \mathrm{e}^{-(\beta + c_0)\tilde{\sigma}}(c_1^2\tilde{\sigma} - c_2)\tilde{\sigma}. \tag{12}$$

Next we compute c_0, c_1 and c_2. By definition of $c(z)$, we have

$$c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2) = G\left(z\gamma + \mathbb{N}\left[1 - \mathrm{e}^{-z(x\mathbf{1}_{\{\tilde{\sigma} > \varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma} \leq \varepsilon\}}) - (\beta + c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2))\tilde{\sigma}}\right]\right).$$

We compute the expansion in z of the right hand-side term of this equality. We set

$$\begin{aligned}
a_0 &= \mathbb{N}\left[1 - \mathrm{e}^{-(\beta + c_0)\tilde{\sigma})}\right] = \psi_\theta^{-1}(\beta + c_0),\\
a_1 &= \gamma + \mathbb{N}\left[\mathrm{e}^{-(\beta + c_0)\tilde{\sigma}}(x\mathbf{1}_{\{\tilde{\sigma} > \varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma} \leq \varepsilon\}} + c_1\tilde{\sigma})\right],\\
a_2 &= \mathbb{N}\left[\mathrm{e}^{-(\beta + c_0)\tilde{\sigma}}(c_2\tilde{\sigma} - (x\mathbf{1}_{\{\tilde{\sigma} > \varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma} \leq \varepsilon\}} + c_1\tilde{\sigma})^2)\right],
\end{aligned}$$

so that standard results on Laplace transform yield

$$c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2) = G\left(a_0 + za_1 + \frac{z^2}{2}a_2 + o(z^2)\right).$$

We deduce that

$$c_0 = G(a_0) = G\left(\mathbb{N}\left[1 - \mathrm{e}^{-(\beta + c_0)\tilde{\sigma}}\right]\right),$$

$$c_1 = a_1 G'(a_0), \tag{13}$$

$$c_2 = a_2 G'(a_0) + a_1^2 G''(a_0) = a_2 G'(a_0) + \frac{c_1^2 G''(a_0)}{G'(a_0)^2}. \tag{14}$$

Using (9) and (7), we have $c_0 = G\left(\psi_\theta^{-1}(\beta + c_0)\right) = \beta + c_0 - \psi(\psi_\theta^{-1}(\beta + c_0))$, that is

$$h_\beta := \beta + c_0 = \psi_\theta(\psi^{-1}(\beta)) \quad \text{and} \quad a_0 = \psi_\theta^{-1}(\beta + c_0) = \psi^{-1}(\beta).$$

Notice that $h_\beta > 0$. And we have, thanks to the second equality of (7),

$$G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}\,\tilde{\sigma}\right] = \frac{\psi_\theta'(\psi^{-1}(\beta)) - \psi'(\psi^{-1}(\beta))}{\psi_\theta'(\psi^{-1}(\beta))} < 1.$$

(This last inequality is equivalent to say that $\dfrac{\partial J}{\partial c}(c(z), z) < 0$ at $z = 0$.) From (13), we get

$$c_1 = G'(\psi^{-1}(\beta))\frac{\gamma + \mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x\mathbf{1}_{\{\tilde{\sigma} > \varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma} \leq \varepsilon\}})\right]}{1 - G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}\,\tilde{\sigma}\right]}, \tag{15}$$

and from (14)

$$c_2 = \frac{-G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}(x\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}}+c_1\tilde\sigma)^2\right]+\dfrac{c_1^2G''(\psi^{-1}(\beta))}{G'(\psi^{-1}(\beta))^2}}{1-G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\tilde\sigma\right]}. \tag{16}$$

We get

$$\mathbb{N}\left[(x\mathcal{N}^\varepsilon+y\mathcal{M}^\varepsilon+\gamma R)^2\,\mathrm{e}^{-\beta\sigma}\right]=c_1^2\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\tilde\sigma^2\right]-c_2\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\tilde\sigma\right], \tag{17}$$

where c_1 and c_2 defined by (15) and (16) are polynomials of respective degree 1 and 2 in x, y and γ. In particular (17) also holds for $x, y, \gamma\in\mathbb{R}$.

In a **fourth step**, we look at asymptotics as ε decreases to 0. Let $\lambda_1, \lambda_2\in\mathbb{R}_+$ and $\gamma=-(\lambda_1+\lambda_2)$. We set

$$x_\varepsilon=\lambda_1/\bar\pi_*(\varepsilon)\quad\text{and}\quad y_\varepsilon=\lambda_2/\varphi(\varepsilon).$$

We recall from Lemma 4.1 in [8], that

$$\lim_{\varepsilon\to0}\frac{1}{\bar\pi_*(\varepsilon)}=0\quad\text{and}\quad\lim_{\varepsilon\to0}\frac{\varepsilon}{\varphi(\varepsilon)}=0. \tag{18}$$

Lemma 7.2 in [1] tells that for any non-negative measurable function F, we have $\mathbb{N}[F(\tilde\sigma)]=\mathbb{N}[\mathrm{e}^{-\theta\sigma}\,F(\sigma)]$. We define

$$\begin{aligned}\Delta_\varepsilon&:=\gamma+\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}(x_\varepsilon\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y_\varepsilon\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}})\right]\\&=-x_\varepsilon\mathbb{N}\left[(1-\mathrm{e}^{-(h_\beta+\theta)\sigma})\mathbf{1}_{\{\sigma>\varepsilon\}}\right]-\varepsilon y_\varepsilon\mathbb{N}\left[(1-\mathrm{e}^{-(h_\beta+\theta)\sigma})\frac{\sigma}{\varepsilon}\mathbf{1}_{\{\sigma\leq\varepsilon\}})\right].\end{aligned}$$

In particular, we have $\Delta_\varepsilon=O\left(\dfrac{1}{\bar\pi_*(\varepsilon)}+\dfrac{\varepsilon}{\varphi(\varepsilon)}\right)$ and from (18) $\lim_{\varepsilon\to0}\Delta_\varepsilon=0$. From (15), we get $c_1=O(\Delta_\varepsilon)=O\left(\dfrac{1}{\bar\pi_*(\varepsilon)}+\dfrac{\varepsilon}{\varphi(\varepsilon)}\right)$. From (16), we also have for some finite constant C independent of ε:

$$|c_2|\leq 2\frac{G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}(x_\varepsilon\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y_\varepsilon\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}})^2\right]}{1-G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\tilde\sigma\right]}+Cc_1^2.$$

Notice that

$$\begin{aligned}\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}(x_\varepsilon\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y_\varepsilon\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}})^2\right]&=\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}(x_\varepsilon^2\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y_\varepsilon^2\tilde\sigma^2\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}})\right]\\&\leq\frac{\lambda_1}{\bar\pi_*(\varepsilon)}\,x_\varepsilon\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}\right]+\frac{\lambda_2\varepsilon}{\varphi(\varepsilon)}\,y_\varepsilon\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde\sigma}\,\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leq\varepsilon\}}\right]\\&=O\left(\frac{1}{\bar\pi_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right).\end{aligned} \tag{19}$$

We deduce that $c_2=O\left(\dfrac{1}{\bar\pi_*(\varepsilon)}+\dfrac{\varepsilon}{\varphi(\varepsilon)}\right)$. Equation (17) implies that

$$\mathbb{N}\left[\left(\lambda_1\frac{\mathcal{N}^\varepsilon}{\bar\pi_*(\varepsilon)}+\lambda_2\frac{\mathcal{M}^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]=O\left(\frac{1}{\bar\pi_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right).$$

As $\sigma \geq \tilde{\sigma}$, we have

$$\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\,\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\,\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}}\right)\mathrm{e}^{-\beta\sigma}\right]\leq\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\,\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\,\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}}\right)\mathrm{e}^{-\beta\tilde{\sigma}}\right]$$
$$=O\left(\frac{1}{\bar{\pi}_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right),$$

where we used (19) for the last equation (with β instead of h_β). Recall that $N^\varepsilon(\theta)=\mathcal{N}^\varepsilon+\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}$ and $M^\varepsilon(\theta)=\mathcal{M}^\varepsilon+\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}}$ and thus

$$\mathbb{N}\left[\left(\lambda_1\frac{N^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{M^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]$$
$$\leq 2\mathbb{N}\left[\left(\lambda_1\frac{\mathcal{N}^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{\mathcal{M}^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]$$
$$+2(\lambda_1^2+\lambda_2^2)\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\,\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\,\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leq\varepsilon\}}\right)\mathrm{e}^{-\beta\sigma}\right]$$

We deduce that

$$\mathbb{N}\left[\left(\lambda_1\frac{N^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{M^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]=O\left(\frac{1}{\bar{\pi}_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right). \tag{20}$$

which, thanks to (18), exactly says that $\lim_{\varepsilon\to 0}\dfrac{N^\varepsilon(\theta)}{\bar{\pi}_*(\varepsilon)}=\lim_{\varepsilon\to 0}\dfrac{M^\varepsilon(\theta)}{\varphi(\varepsilon)}=R$ in $L^2(\mathbb{N}[\mathrm{e}^{-\beta\sigma}\,\cdot])$.

3.2. **Proof of Proposition 1.2.** Recall that, in the stable case, we have

$$\bar{\pi}_*(\varepsilon)=\frac{1}{\Gamma(1-1/\alpha)}\varepsilon^{-1/\alpha}\qquad\text{and}\qquad\varphi(\varepsilon)=\frac{1}{(\alpha-1)\Gamma(1-1/\alpha)}\varepsilon^{1-1/\alpha}.$$

Therefore $\varepsilon_n=n^{-2\alpha}$, $n\geq 1$, satisfies $\displaystyle\sum_{n\geq 1}\frac{1}{\bar{\pi}_*(\varepsilon_n)}+\frac{\varepsilon_n}{\varphi(\varepsilon_n)}<\infty$. The series with general term given by the left hand-side of (20) with $\varepsilon=\varepsilon_n$ is convergent. This implies that $\mathbb{N}$-a.e (and $\mathbb{N}_1$-a.s.)

$$\lim_{n\to\infty}\frac{N^{\varepsilon_n}(\theta)}{\bar{\pi}_*(\varepsilon_n)}=\lim_{n\to\infty}\frac{M^{\varepsilon_n}(\theta)}{\varphi(\varepsilon_n)}=R(\theta).$$

Since $N^\varepsilon(\theta)$ is a non-increasing function of ε, we get that for any $\varepsilon\in[(n+1)^{-2\alpha},n^{-2\alpha}]$, we have

$$\frac{n^2}{(n+1)^2}n^{-2}N^{n^{-2\alpha}}(\theta)\leq\varepsilon^{1/\alpha}N^\varepsilon(\theta)\leq\frac{(n+1)^2}{n^2}(n+1)^{-2}N^{(n+1)^{-2\alpha}}(\theta).$$

Hence we deduce that $\mathbb{N}$-a.e. or $\mathbb{N}_1$-a.s., $\lim_{\varepsilon\to\,0}\varepsilon^{1/\alpha}N^\varepsilon(\theta)=R(\theta)/\Gamma(1-\alpha^{-1})$.

The proof for $M^\varepsilon(\theta)$ is similar, as $M^\varepsilon(\theta)$ is a non-decreasing function of ε.

4. Law of $R(\theta)$

Lemma 4.1. *Let $\beta\geq 0$, $\gamma\leq 0$. We have*

$$\mathbb{N}\left[1-\mathrm{e}^{-\beta\sigma-\gamma R(\theta)}\right]=v$$

where v is the unique non-negative root of

$$\beta+\psi(\gamma+\theta+v)=\psi(v+\theta)+\psi(v+\gamma). \tag{21}$$

Remark 4.2. For the limit case $\psi(\lambda) = \lambda^2$ (which is excluded here), we get the unique non-negative root of (21) is $v = \sqrt{\lambda + 2\gamma\theta}$. This would implies $R(\theta) = 2\theta\sigma$ $\mathbb{N}$-a.e. and $R(\theta) = 2\theta$ $\mathbb{N}_1$-a.s. This agrees with the result in [6], where the limit which appears for (2) is a.s. equal to 2θ.

Proof. Take $x = y = 0$ in (11), integrate w.r.t. $\mathbb{N}$ and use (7) to get

$$\mathbb{N}\left[1 - \mathrm{e}^{-\gamma R(\theta) - \beta\sigma}\right] = \mathbb{N}\left[1 - \mathrm{e}^{-(\beta+c)\tilde{\sigma}(\theta)}\right] = \psi_\theta^{-1}(\beta + c),$$

where c is the unique root of $c = H_{(0,0,\gamma)}(c)$ that is of $c = G(\gamma + \psi_\theta^{-1}(\beta + c))$. If we set $v = \psi_\theta^{-1}(\beta + c)$, we have that v is the unique non-negative root of the equation $G(\gamma + v) = \psi_\theta(v) - \beta$, that is (21). □

5. Appendix

Let $\alpha \in (1, 2)$. Recall from [1] Corollary 9.3 or [13] that the fragmentation is self similar with index $1/\alpha$ and dislocation measure given by

$$\int_{\mathcal{S}^\downarrow} F(x)\nu_1(dx) = \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)}\mathbb{E}[S_1\ F((\Delta S_t/S_1, t \le 1))],$$

where F is any non-negative measurable function on $\mathcal{S}^\downarrow$, and $(\Delta S_t, t \geq 0)$ are the jumps of a stable subordinator $S = (S_t, t \geq 0)$ of Laplace exponent $\psi^{-1}(\lambda) = \lambda^{1/\alpha}$, ranked by decreasing size.

In this section we shall compute the functions f_b, φ_b and g_b defined in [6] and recalled in Remark 1.4 for the self-similar fragmentation at nodes.

Lemma 5.1. *We have* $f_b(\varepsilon) = \dfrac{1}{\Gamma(1+1/\alpha)}\left(\dfrac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha}$.

Proof. The Lévy measure of S is given by $\pi_*(dr) = \dfrac{1}{\alpha\Gamma(1-1/\alpha)}\dfrac{dr}{r^{1+1/\alpha}}dr$. For $\beta \in (0, 1)$, we have $\displaystyle\int_{(0,\infty)} \frac{dy}{y^{1+\beta}}(1 - \mathrm{e}^{-y\lambda}) = \lambda^\beta\frac{\Gamma(1-\beta)}{\beta}$. We deduce that

$$\mathbb{E}[S_1^\beta] = \frac{\beta}{\Gamma(1-\beta)}\mathbb{E}\left[\int_0^\infty \frac{dy}{y^{1+\beta}}(1 - \mathrm{e}^{-yS_1})\right] = \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty \frac{dy}{y^{1+\beta}}(1 - \mathrm{e}^{-y^{1/\alpha}}) = \frac{\Gamma(1-\alpha\beta)}{\Gamma(1-\beta)}.$$

Standard computation for Poisson measure yield

$$\begin{aligned}
f_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow} \sum_{i=1}^\infty x_i \mathbf{1}_{\{x_i<\varepsilon\}} \nu_1(dx) \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \sum_{t\le 1} \frac{\Delta S_t}{S_1} \mathbf{1}_{\{\Delta S_t < \varepsilon S_1\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\sum_{t\le 1} \Delta S_t \mathbf{1}_{\{\Delta S_t < \varepsilon (S_1-\Delta S_t)/(1-\varepsilon)\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\int \pi_*(dr) r \mathbf{1}_{\{r<\varepsilon S_1/(1-\varepsilon)\}} \right] \\
&= \frac{\alpha}{\Gamma(2-\alpha)} \mathbb{E}[S_1^{1-1/\alpha}] \left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha} \\
&= \frac{1}{\Gamma(1+1/\alpha)} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha}.
\end{aligned}$$

□

Lemma 5.2. *We have* $\lim_{\varepsilon\to 0} \varepsilon^{1/\alpha}\varphi_b(\varepsilon) = \dfrac{\alpha-1}{\Gamma(1+1/\alpha)}$.

Proof. We have

$$\begin{aligned}
\varphi_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow} (\sum_{i=1}^\infty \mathbf{1}_{\{x_i>\varepsilon\}} - 1) \nu_1(dx) \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \sum_{t\le 1} \mathbf{1}_{\{\Delta S_t > \varepsilon S_1\}} - S_1 \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\sum_{t\le 1} (S_1-\Delta S_t) \mathbf{1}_{\{\Delta S_t > \varepsilon \frac{S_1-\Delta S_t}{1-\varepsilon}\}} - \Delta S_t \mathbf{1}_{\{\Delta S_t \le \varepsilon \frac{S_1-\Delta S_t}{1-\varepsilon}\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \int \pi_*(dr) \mathbf{1}_{\{r>\varepsilon S_1/(1-\varepsilon)\}} \right] - f_b(\varepsilon) \\
&= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1^{1-1/\alpha} \right] \left(\frac{\varepsilon}{1-\varepsilon}\right)^{-1/\alpha} - f_b(\varepsilon) \\
&= \frac{\alpha-1}{\Gamma(1+1/\alpha)} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{-1/\alpha} - f_b(\varepsilon)
\end{aligned}$$

□

Lemma 5.3. *The limit* $\lim_{\varepsilon\to 0} \dfrac{g_b(\varepsilon)}{f_b(\varepsilon)^2}$ *exists and belongs to* $(0,\infty)$.

Proof. We have

$$\begin{aligned}
g_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^{\infty} x_i \mathbf{1}_{\{x_i<\varepsilon\}} \right)^2 \nu_1(dx) \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \left(\sum_{t\leq 1} \frac{\Delta S_t}{S_1} \mathbf{1}_{\{\Delta S_t < \varepsilon S_1\}} \right)^2 \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \Big(\mathbb{E}\Big[\sum_{t\leq 1} \frac{(\Delta S_t)^2}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}} \Big] \\
&\qquad + \mathbb{E}\Big[\sum_{t\leq 1, s\leq 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1, \Delta S_s < \varepsilon S_1\}} \Big] \Big).
\end{aligned}$$

For the first term, we get

$$\begin{aligned}
\mathbb{E}\Big[\sum_{t\leq 1} \frac{(\Delta S_t)^2}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}} \Big] &\leq \mathbb{E}\Big[\sum_{t\leq 1} \frac{(\Delta S_t)^2}{S_1 - \Delta S_t} \mathbf{1}_{\{\Delta S_t<\varepsilon (S_1-\Delta S_t)/(1-\varepsilon)\}} \Big] \\
&= \mathbb{E}\left[\frac{1}{S_1} \int \pi_*(dr) r^2 \mathbf{1}_{\{r<\varepsilon S_1/(1-\varepsilon)\}} \right] \\
&= \frac{1}{(2\alpha-1)\Gamma(1-1/\alpha)} \mathbb{E}[S_1^{1-1/\alpha}] \left(\frac{\varepsilon}{1-\varepsilon} \right)^{2-1/\alpha} \\
&= o(\varepsilon^{2-2/\alpha}).
\end{aligned}$$

For the second term, we notice that for $r, s, S \in \mathbb{R}_+$

$$\begin{aligned}
\{r \leq \varepsilon S/(1-\varepsilon), v \leq \varepsilon S/(1-\varepsilon)\} &\subset \{r \leq \varepsilon(S+r+v), v \leq \varepsilon(S+r+v)\} \\
&\subset \{r \leq \varepsilon S/(1-2\varepsilon), v \leq \varepsilon S/(1-2\varepsilon)\}.
\end{aligned}$$

And we get

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{t\leq 1, s\leq 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1, \Delta S_s < \varepsilon S_1\}} \Big] \\
&\qquad \leq \mathbb{E}\Big[\sum_{t\leq 1, s\leq 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1 - \Delta S_t - \Delta S_s} \mathbf{1}_{\{\Delta S_t < \varepsilon \frac{S_1-\Delta S_t-\Delta S_s}{1-2\varepsilon}, \Delta S_s < \varepsilon \frac{S_1-\Delta S_t-\Delta S_s}{1-2\varepsilon}\}} \Big] \\
&\qquad = \mathbb{E}\left[\frac{1}{S_1} \left(\int \pi_*(dr) r \mathbf{1}_{\{r<\varepsilon S_1/(1-2\varepsilon)\}} \right)^2 \right] \\
&\qquad = c_\alpha \left(\frac{\varepsilon}{1-2\varepsilon} \right)^{2-2/\alpha},
\end{aligned}$$

with $c_\alpha = \dfrac{\Gamma(3-\alpha)}{(\alpha-1)^2\Gamma(2/\alpha)\Gamma(1-1/\alpha)^2}$, as well as

$$\begin{aligned}
\mathbb{E}\Big[\sum_{t\leq 1, s\leq 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1} \mathbf{1}_{\{\Delta S_t < \varepsilon S_1, \Delta S_s < \varepsilon S_1\}} \Big] & \\
\geq \mathbb{E}\Big[\sum_{t\leq 1, s\leq 1, s\neq t} \frac{\Delta S_t \Delta S_s}{(S_1 - \Delta S_t - \Delta S_s)\frac{1+2\varepsilon}{1-\varepsilon}} \mathbf{1}_{\{\Delta S_t < \varepsilon \frac{S_1 - \Delta S_t - \Delta S_s}{1-\varepsilon}, \Delta S_s < \varepsilon \frac{S_1 - \Delta S_t - \Delta S_s}{1-\varepsilon}\}} \Big] & \\
= \frac{1-\varepsilon}{1+2\varepsilon} \mathbb{E}\left[\frac{1}{S_1} \left(\int \pi_*(dr) r \mathbf{1}_{r < \varepsilon S_1/(1-\varepsilon)\}} \right)^2 \right] & \\
= c_\alpha \frac{1-\varepsilon}{1+2\varepsilon} \left(\frac{\varepsilon}{1-\varepsilon} \right)^{2-2/\alpha}. &
\end{aligned}$$

In particular, we have that $g_b(\varepsilon) = c_\alpha \varepsilon^{2-2/\alpha}(1+o(1))$. We deduce that $\lim_{\varepsilon\to 0} \dfrac{g_b(\varepsilon)}{f_b(\varepsilon)^2} \in (0,\infty)$. □

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